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Two-loop QCD correction to differential semi-leptonic $b \to u$ decays in the shape-function region

M. Beneke[a,b], T. Huber[a], and Xin-Qiang Li[a,1]

[a] *Institut für Theoretische Physik E, RWTH Aachen University, D–52056 Aachen, Germany*

[b] *Institut für Theoretische Physik, Universität Zürich, CH – 8057 Zürich, Switzerland*

Abstract

We calculate the two-loop QCD correction to the form factors of on-shell b-quark decay to an energetic massless quark, which constitutes the last missing piece required for an $\mathcal{O}(\alpha_s^2)$ determination of $|V_{ub}|$ from inclusive semi-leptonic $\bar{B} \to X_u \ell \bar{\nu}$ decays in the shape-function region.

[1] Alexander-von-Humboldt Fellow

1 Introduction

The strength of $b \to u$ transitions, measured by the CKM element $|V_{ub}|$, calibrates one of the sides of the unitarity triangle and is therefore an important input to flavour physics. Exclusive semi-leptonic $\bar{B} \to M\ell\bar{\nu}$ decays (with $M = \pi, \rho, \ldots$) provide direct access to the $b \to u$ transition, but require knowledge of the heavy-to-light meson form factors to extract $|V_{ub}|$. Inclusive semi-leptonic heavy quark decays on the other hand, can be calculated in perturbation theory. However, the need to separate $\bar{B} \to X_u\ell\bar{\nu}$ from an overwhelming charm final-state background requires cuts on the differential decay spectra, which in one way or another constrain the hadronic final state to have small invariant mass and large energy. The final state distribution then depends on a non-perturbative function describing the light-cone residual momentum distribution of the b-quark in the $\bar{B}$ meson, the shape function [1, 2]. Somewhat unfortunately, the two methods currently result in values of $|V_{ub}|$ that seem to systematically differ, with the exclusive method favouring smaller $|V_{ub}|$. This provides motivation and urgency to improving the theoretical accuracy of both methods.

The differential distributions in inclusive semi-leptonic decay are calculated from the discontinuity

$$W^{\mu\nu} = \frac{1}{\pi}\,\mathrm{Im}\left[\langle \bar{B}(p_B)|\, i\int d^4x\, e^{iq\cdot x}\, T(j^{\dagger\mu}(0)j^{\nu}(x))|\bar{B}(p_B)\rangle\right] \tag{1}$$

of the forward matrix element of a correlation function of the weak interaction current $j^{\mu} = \bar{u}\gamma^{\mu}(1-\gamma_5)b$. In the shape-function region the structure functions into which the hadronic tensor $W^{\mu\nu}$ can be decomposed, can be organized into the factorized expression [3]

$$W_k = \sum_{i,j} a_k^{ij}\, C_i(n_-\cdot p)\, C_j(n_-\cdot p) \int d\omega\, J(p_\omega^2)\, S(\omega) \tag{2}$$

at leading power in the $1/m_b$ expansion. The factors C_iC_j, J, S arise from the different scales m_b, $\sqrt{m_b\Lambda_{\mathrm{QCD}}}$ and Λ_{QCD}, respectively, such that C_i and J can be computed in QCD perturbation theory, while the B meson shape function is non-perturbative. (a_k^{ij} are numerical constants.) Eq. (2) has been worked out at order $\mathcal{O}(\alpha_s)$ in [4, 5] and forms the basis of the inclusive $|V_{ub}|$ analysis performed in [6]. A summary of other inclusive $|V_{ub}|$ analyses and recent experimental results based on these methods is given in [7]. At the two-loop order, the jet function J and the partonic shape function of the b quark have already been calculated [8,9]. Here we compute the two-loop hard matching coefficients C_i associated with the form factors of on-shell b-quark decay into an energetic massless quark and provide a numerical estimate of the new term. All items for a $\mathcal{O}(\alpha_s^2)$ determination of $|V_{ub}|$ from inclusive semi-leptonic decay are now in place, which should remove most of the perturbative theoretical uncertainty. A detailed phenomenological analysis is, however, beyond the scope of the current work.

While this paper was in preparation, the calculation reported here has also been completed by Bonciani and Ferroglia [10]. The results have been compared prior to publication and complete agreement has been found. On the day of submission of the present work, the paper [11] on the same topic has appeared.

2 Structure of the calculation

2.1 Set-up of the matching calculation

The calculation of the short-distance coefficients C_i in (2) amounts to matching the current $\bar{u}\gamma^\mu(1-\gamma_5)b$ to a set of leading-power currents in soft-collinear effective theory (SCET) [12–14]. One-loop results for the C_i have been obtained in this framework in [12, 15]. The current is represented in SCET by

$$[\bar{u}\gamma^\mu(1-\gamma_5)b](0) = \int d\hat{s} \sum_i \widetilde{C}_i(\hat{s})\, (\bar{\xi}W_c)(sn_+)\Gamma_i^\mu h_v(0) + \ldots , \qquad (3)$$

where ξ is the collinear up-quark field in SCET, h_v the static heavy-quark field, and W_c a collinear Wilson line. (We use notation as defined in more detail in [15].) There are three independent Dirac matrices that can appear on the right-hand side, which we choose as

$$\Gamma_1^\mu = \gamma^\mu(1-\gamma_5)\,, \qquad \Gamma_2^\mu = v^\mu(1+\gamma_5)\,, \qquad \Gamma_3^\mu = n_-^\mu(1+\gamma_5)\,. \qquad (4)$$

The ellipses in (3) denote higher-dimensional operators, which are not relevant to the present work, and $\hat{s} \equiv sm_b$. SCET makes extensive use of two light-like vectors n_+^μ, n_-^μ with $n_+ \cdot n_- = 2$, with respect to which four-vectors are decomposed as

$$p^\mu = n_+ \cdot p\,\frac{n_-^\mu}{2} + n_- \cdot p\,\frac{n_+^\mu}{2} + p_\perp^\mu, \qquad (5)$$

with $n_- \cdot p_\perp = n_+ \cdot p_\perp = 0$. The collinear field ξ describes modes which have $n_+ \cdot p$ large, of order m_b. To define the heavy-quark field, we choose a frame such that $v^\mu = (n_+^\mu + n_-^\mu)/2$, that is $n_+ \cdot v = n_- \cdot v = 1$. The factorization formula (2) for the differential decay distributions makes use of the momentum-space short-distance coefficient rather than the position-space expression appearing in the convolution in (3). The momentum space coefficient functions are related to those defined above by

$$C_i(u) = \int d\hat{s}\, e^{iu\hat{s}}\, \widetilde{C}_i(\hat{s}), \qquad (6)$$

where the new variable $u \in [0, 1]$ equals the momentum fraction $n_+ \cdot p/m_b$ of the external up-quark line in a momentum-space Feynman diagram.

The actual matching calculation is also done in momentum space and yields the momentum-space coefficient functions directly. To this end we consider the matrix element of the current between a bottom quark of mass m_b and momentum p_b and a massless up-quark with momentum p. Both quarks are considered on-shell, hence $p_b^2 = m_b^2$ and

$p^2 = 0$. The absence of any perturbative infrared scale implies that the SCET loop diagrams are scaleless, and the SCET matrix element is given by its tree-level expression multiplied by the universal renormalization factor for the SCET currents $(\bar{\xi} W_c)\Gamma_i^\mu h_v$:

$$\langle u(p)|(\bar{\xi} W_c)(sn_+)\Gamma_i^\mu h_v(0)|b(p_b)\rangle = e^{isn_+\cdot p}\, Z_J\, \bar{u}_{n_-}\, \Gamma_i^\mu\, u_v\,. \tag{7}$$

Thus, calculating the QCD matrix element in dimensional regularization yields directly the dimensionally regularized short-distance coefficient $C_i(u)$. To this end, we first decompose the matrix element of the hadronic current into three form factors according to

$$\begin{aligned}\langle u(p)|\bar{u}\gamma^\mu(1-\gamma_5)b|b(p_b)\rangle \;=\; & F_1(u)\;\bar{u}(p)\gamma^\mu(1-\gamma_5)u(p_b) + F_2(u)\;\bar{u}(p)\frac{p_b^\mu}{m_b}(1+\gamma_5)u(p_b) \\ & + F_3(u)\;\bar{u}(p)\frac{m_b\, p^\mu}{p_b\cdot p}(1+\gamma_5)u(p_b)\,. \end{aligned} \tag{8}$$

The form factors $F_i(u)$, $i = 1, 2, 3$ can only depend on the dimensionless variable $u \equiv 2\, p_b \cdot p/m_b^2$ and logarithms of μ^2/m_b^2, where μ is at once the renormalization scale of the strong coupling and the infrared factorization scale, since the on-shell heavy-to-light form factors contain soft and collinear divergences. Only F_1 is non-zero at tree-level, $F_1 = 1 + \mathcal{O}(\alpha_s)$, while $F_{2,3} = \mathcal{O}(\alpha_s)$. Identifying $p_b^\mu = m_b v^\mu$ and $n_-^\mu = m_b p^\mu/(p_b \cdot p)$, we see that u equals the variable u defined in (6). At leading order in the heavy-quark expansion the spinors of the quark fields equal $\bar{u}(p) = \bar{u}_{n_-}$, $u(p_b) = u_v$, where the collinear and heavy quark spinors satisfy $\not{n}_- u_{n_-} = 0$ and $\not{v} u_v = u_v$, respectively. Eq. (8) then becomes

$$\langle u(p)|\bar{u}\gamma^\mu(1-\gamma_5)b|b(p_b)\rangle \;=\; \sum_{i=1}^{3} F_i(u)\;\bar{u}_{n_-}\,\Gamma_i^\mu\, u_v, \tag{9}$$

while using (7) the matrix element of the right-hand side of (3) equals

$$\sum_{i=1}^{3} Z_J\, C_i(n_+\cdot p/m_b)\;\bar{u}_{n_-}\,\Gamma_i^\mu\, u_v, \tag{10}$$

so we simply have $C_i(u) = Z_J^{-1} F_i(u)$, $i = 1, 2, 3$. In the following we briefly describe the method of calculating the two-loop QCD correction to the F_i.

2.2 Calculational methods

The computation of the two-loop QCD vertex corrections to semi-leptonic $b \to u$ decays involves the evaluation of the diagrams shown in Fig. 1. We work in dimensional regularization with $D = 4 - 2\epsilon$, where UV and IR (soft and collinear) divergences appear as poles of up to the fourth order in ϵ. For the matrix γ_5 we adopt the naive dimensional regularization (NDR) scheme with anticommuting γ_5. Furthermore, we treat all quarks

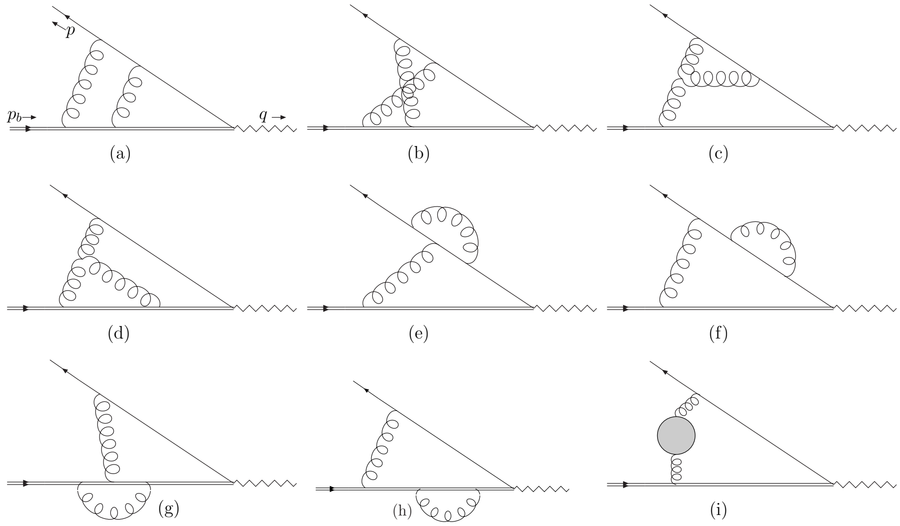

Figure 1: Two-loop diagrams needed for the calculation. Double straight lines stand for massive quarks of mass m_b, whereas single ones stand for massless quarks. The filled circle in the last diagram represents the complete one-loop gluon self-energy with n_f massless quarks (including the charm) and the massive bottom quark.

except the bottom quark as massless. We checked that it is possible to include the effects of a non-zero charm mass analytically. However, since in the two-loop calculation of the jet and shape function [8, 9] the charm mass is neglected, we also set it to zero in the present work.

The amplitude of the diagrams is reduced by techniques that have become standard in multi-loop calculations. We apply a Passarino – Veltman [16] reduction of the vector and tensor integrals. The Dirac and color algebra is then performed by means of an in-house Mathematica routine. The dimensionally regularized scalar integrals are further reduced to a small set of master integrals (depicted in Fig. 4 in the Appendix) using the Laporta algorithm [17, 18] based on integration-by-parts (IBP) identities [19, 20]. To this end we use the Maple package AIR [21].

The techniques we apply during the evaluation of the master integrals are manifold. The easier integrals can be written in a closed form in terms of Γ-functions and hypergeometric functions and subsequently expanded in ϵ with the package `HypExp` [22, 23]. In more complicated cases we derive Mellin-Barnes representations by means of the package AMBRE [24]. We perform the analytic continuation to $\epsilon = 0$ with the package MB [25], which is also used for numerical cross-checks. We then apply Barnes' lemmas and the theorem of residues on the multiple Mellin-Barnes integrals, and insert integral representations of hypergeometric functions as well as ψ-functions and Euler's B-function where appropriate. As a third technique we apply the method of differential equations [26–28] and evaluate the boundary condition with the Mellin-Barnes technique. This renders, for instance, a three-dimensional MB representation in the case of the crossed six-line master integral of Fig. 4(g) at $u = 1$. Eventually, the master integrals are evaluated as

Laurent series in ϵ, with expansion coefficients of argument u expressed analytically in logarithms and polylogarithms of increasing weight. The maximum weight which appears in our calculation is four, and we can express all functions but one in terms of ordinary polylogarithms. The function that cannot be expressed in terms of ordinary polylogarithms is the harmonic polylogarithm [29] $\mathrm{HPL}(\{-2,2\},1-u)$. We left, however, also a second function, $\mathrm{HPL}(\{-1,2\},1-u)$, in the HPL notation since its expression in terms of ordinary polylogarithms is rather complicated [30]. The master integrals have been calculated already in [31] and used in [32]. We find (almost) perfect agreement on the expressions in [31]. Moreover, we improve the numerical accuracy of one of the boundary conditions, see Appendix A.

3 Renormalization

After the pure two-loop calculation the result still contains divergences of UV as well as IR (soft and collinear) nature. The former divergences are cancelled after addition of the UV counterterms, the latter disappear after inclusion of the jet and shape function contributions, since the partonic differential decay distributions are infrared-finite. In the following we also present the pole parts of the form factors F_i. Subtracting the IR poles as discussed in Sec. 5 leads to the $\overline{\mathrm{MS}}$ definition of the SCET current.

3.1 UV renormalization

In performing the UV renormalization we adopt the on-shell scheme for the heavy quark mass as well as for the heavy and light quark field. The strong coupling α_s on the other hand is renormalized in the $\overline{\mathrm{MS}}$ scheme. The respective renormalization constants read [33–35]

$$Z_m^{\mathrm{os}} = 1 - C_F\,\frac{g_0^2 (m^2)^{\frac{D-4}{2}}}{(4\pi)^{D/2}}\,\frac{(D-1)\Gamma(2-\frac{D}{2})}{(D-3)}\,, \tag{11}$$

$$\begin{aligned}
Z_h^{\mathrm{os}} = {} & 1 - C_F\,\frac{g_0^2 (m^2)^{\frac{D-4}{2}}}{(4\pi)^{D/2}}\,\frac{(D-1)\Gamma(2-\frac{D}{2})}{(D-3)} \\
& + \frac{g_0^4 (m^2)^{D-4}}{(4\pi)^D}\,\Gamma^2(3-\frac{D}{2})\left\{C_F^2\left[\frac{18}{(D-4)^2} - \frac{51}{2(D-4)}\right.\right. \\
& \left. + \frac{433}{8} - 13\pi^2 + 16\pi^2\ln(2) - 24\zeta_3 + \mathcal{O}(D-4)\right] + C_F\,C_A\left[-\frac{22}{(D-4)^2}\right. \\
& \left. + \frac{101}{2(D-4)} - \frac{803}{8} + 5\pi^2 - 8\pi^2\ln(2) + 12\zeta_3 + \mathcal{O}(D-4)\right] \\
& + C_F\,n_f\,t_f\left[\frac{8}{(D-4)^2} - \frac{18}{(D-4)} + \frac{4\pi^2}{3} + \frac{59}{2} + \mathcal{O}(D-4)\right]
\end{aligned}$$

$$+C_F\,t_f\left[\frac{16}{(D-4)^2}-\frac{38}{3(D-4)}-\frac{16\pi^2}{3}+\frac{1139}{18}+\mathcal{O}(D-4)\right]\Bigg\}\ . \tag{12}$$

Here g_0 is the bare QCD coupling. $C_F=(N^2-1)/(2\,N)$ and $C_A=N$ are the Casimir operators of the fundamental and adjoint representation of SU(N), respectively, and $t_f=1/2$ denotes the normalization of the trace of two fundamental generators. n_f stands for the number of massless quarks, and we set the number of heavy quarks of mass m to unity throughout the paper. The renormalization constant for a massless quark field in the on-shell scheme receives corrections only from two-loop and higher due to diagrams with massive quark loops. At two loops only a single diagram contributes and the Z factor assumes the following closed form

$$Z_l^{\rm os}=1+2\,C_F\,t_f\,\frac{g_0^4(m^2)^{D-4}}{(4\pi)^D}\,\frac{(D-1)\Gamma(4-\frac{D}{2})\Gamma(-\frac{D}{2})}{(D-5)(D-7)}\ . \tag{13}$$

The renormalization constant Z_α of the strong coupling in the $\overline{\rm MS}$ scheme reads

$$Z_\alpha=1+\frac{\alpha_s}{4\pi}\left[\frac{11}{3}C_A-\frac{4}{3}\,t_f\,(n_f+1)\right]\frac{2}{(D-4)}+\mathcal{O}(\alpha_s^2)\ . \tag{14}$$

The renormalization of the quark fields amounts to mere multiplications. Expanding the bare and renormalized amplitude as well as the renormalization constants according to

$$\mathcal{A}_{\rm bare}=\mathcal{A}^{(0)}+g_0^2\,\mathcal{A}^{(1)}_{\rm bare}+g_0^4\,\mathcal{A}^{(2)}_{\rm bare}+\mathcal{O}(g_0^6)\ , \tag{15}$$

$$\mathcal{A}_{\rm ren}=\mathcal{A}^{(0)}+g_0^2\,\mathcal{A}^{(1)}_{\rm ren}+g_0^4\,\mathcal{A}^{(2)}_{\rm ren}+\mathcal{O}(g_0^6)\ , \tag{16}$$

$$Z_i=1+g_0^2\,\delta Z_i^{(1)}+g_0^4\,\delta Z_i^{(2)}+\mathcal{O}(g_0^6)\ ,\qquad i=m,h,l, \tag{17}$$

we find for the renormalized amplitude

$$\mathcal{A}^{(1)}_{\rm ren}=\mathcal{A}^{(1)}_{\rm bare}+\frac{1}{2}\,\delta Z_h^{(1)}\,\mathcal{A}^{(0)}\ , \tag{18}$$

$$\mathcal{A}^{(2)}_{\rm ren}=\mathcal{A}^{(2)}_{\rm bare}+\frac{1}{2}\,\delta Z_h^{(1)}\,\mathcal{A}^{(1)}_{\rm bare}+\left[\frac{1}{2}\,\delta Z_h^{(2)}-\frac{1}{8}\,(\delta Z_h^{(1)})^2+\frac{1}{2}\,\delta Z_l^{(2)}\right]\mathcal{A}^{(0)}\ . \tag{19}$$

The bare coupling $\alpha_s^0=g_0^2/(4\pi)$ is then renormalized simply by the procedure $\alpha_s^0=Z_\alpha\,\alpha_s\,\tilde{\mu}^{2\epsilon}$ with $\tilde{\mu}^2=\mu^2\exp(\gamma_E-\ln 4\pi)$. This can also be seen from the way we present our results in (22), where Z_α accounts for the renormalization of the coupling constant.

The only non-trivial contribution to the UV renormalization is therefore the one-loop diagram in Fig. 2. To this end, only that part of the counterterm Feynman rule that contains the one-loop correction $\delta Z_m^{(1)}$ to Z_m has to be inserted, and the contribution has to be added to the RHS of Eq. (19).

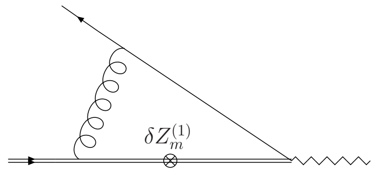

Figure 2: One-loop counterterm diagram for mass renormalization

3.2 IR subtraction, jet and shape function contribution

Since the partonic structure functions W_i in (2) are infrared-finite, the $1/\epsilon$ poles of the two-loop coefficients C_i or, equivalently, F_i can be constructed independently using the two-loop expressions of the jet and shape functions [8, 9] and the one-loop coefficients F_i including their $\mathcal{O}(\epsilon)$ parts (given below). To check our calculation, we perform the convolution $\int d\omega\, J(p_\omega^2)\, S(\omega)$ of the unrenormalized jet and shape function with $\mathcal{O}(\alpha_s^2)$ accuracy and determine the infrared pole part of each F_i individually by forming appropriate combinations of i, j. We find complete agreement with the poles obtained in the direct two-loop calculation of the F_i.

In order to verify the pole cancellation in (2), one must remember that the calculation of the short-distance coefficients C_i is performed in a theory with five active flavours (the fifth being the bottom quark), hence α_s that appears above is $\alpha_s^{(5)}$, while the jet and shape function are computed in four-flavour SCET. In combining C_i, J and S according to (2), the perturbative expansion of C_i must be expressed in terms of $\alpha_s^{(4)}$. The D-dimensional relation between the renormalized couplings required for this purpose reads to one-loop accuracy

$$\alpha_s^{(5)} = \xi_{45}\, \alpha_s^{(4)}\,, \quad \text{where} \quad \xi_{45} = 1 + \frac{\alpha_s^{(4)}}{4\pi}\, \frac{4\, t_f}{3} \left[e^{\gamma_E \epsilon}\, \Gamma(\epsilon) \left(\frac{m_b^2}{\mu^2} \right)^{-\epsilon} - \frac{1}{\epsilon} \right] \tag{20}$$

and the couplings are evaluated at the scale μ.

The $\overline{\text{MS}}$ renormalized coefficient functions are obtained from $C_i = Z_J^{-1} F_i$ (see above), where [12]

$$Z_J = 1 + \frac{\alpha_s^{(4)} C_F}{4\pi} \left\{ -\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[-\ln\left(\frac{\mu^2}{u^2 m_b^2} \right) - \frac{5}{2} \right] \right\} + \mathcal{O}(\alpha_s^2) \tag{21}$$

at one loop, and the two-loop renormalization factor can be obtained from the result of our calculation by requiring that C_1 is free of $1/\epsilon$ poles at $\mathcal{O}(\alpha_s^2)$.

4 Results

We present the result for the short-distance coefficients (form factors) in the form

$$F_i = F_i^{(0)} + \frac{Z_\alpha \alpha_s}{4\pi}\, F_i^{(1)} + \frac{Z_\alpha^2\, \alpha_s^2}{(4\pi)^2}\, F_i^{(2)} + \mathcal{O}(\alpha_s^3)\ , \tag{22}$$

with

$$F_1^{(0)} = 1\,, \qquad F_2^{(0)} = 0\,, \qquad F_3^{(0)} = 0\,. \tag{23}$$

The factor Z_α is given in (14), and Z_α as well as α_s in (22) refer to a theory with five active quark flavors, $n_f = 4$ massless ones and one massive one of mass m_b. Note that we expressed the series in terms of $Z_\alpha \alpha_s$ rather than the renormalized coupling α_s. Thus, the $F_i^{(2)}$ contain $1/\epsilon$ poles that are cancelled by charge renormalization as well as the IR poles discussed above.

4.1 One-loop results

The one-loop hard matching coefficients are available in the literature through order $\mathcal{O}(\epsilon^0)$ [5, 12, 15], but the two-loop analysis of the structure functions W_i requires them up to order $\mathcal{O}(\epsilon^2)$. Here we summarize the corresponding expressions for the form factors.

$$\begin{aligned} F_1^{(1)} = C_F \Big\{ & -\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left[f_{-1}(u) - L\right] - \frac{L^2}{2} + L\, f_{-1}(u) + f_0(u) \\ & +\epsilon\left[-\frac{L^3}{6} + \frac{1}{2}\, L^2\, f_{-1}(u) + L\, f_0(u) + f_1(u)\right] \\ & +\epsilon^2\left[-\frac{L^4}{24} + \frac{1}{6}\, L^3\, f_{-1}(u) + \frac{1}{2} L^2\, f_0(u) + L\, f_1(u) + f_2(u)\right]\Big\}\,, \end{aligned} \tag{24}$$

$$F_2^{(1)} = C_F\left\{g_0(u) + \epsilon\left[g_1(u) + L\, g_0(u)\right] + \epsilon^2\left[g_2(u) + L\, g_1(u) + \frac{L^2}{2}\, g_0(u)\right]\right\}\,, \tag{25}$$

$$F_3^{(1)} = C_F\left\{h_0(u) + \epsilon\left[h_1(u) + L\, h_0(u)\right] + \epsilon^2\left[h_2(u) + L\, h_1(u) + \frac{L^2}{2}\, h_0(u)\right]\right\}\,, \tag{26}$$

with

$$L \equiv \ln\left(\frac{\mu^2}{m_b^2}\right)\,, \tag{27}$$

and

$$f_{-1}(u) = 2\ln(u) - \frac{5}{2}\,, \tag{28}$$

$$f_0(u) = -2\ln^2(u) + 2\ln(u)\ln(1-u) + 2\,\mathrm{Li}_2(u) + \frac{\ln(u)}{u-1} + 3\ln(u) - \frac{5\pi^2}{12} - 6\,, \tag{29}$$

$$\begin{aligned} f_1(u) = {} & \frac{4}{3}\ln^3(u) - 2\ln^2(u)\ln(1-u) - \frac{\ln^2(u)}{u-1} - 3\ln^2(u) + \frac{\ln(1-u)\ln(u)}{u-1} \\ & +3\ln(u)\ln(1-u) + \frac{4\ln(u)}{u-1} + \frac{5\pi^2}{6}\,\ln(u) + 8\ln(u) + \frac{\mathrm{Li}_2(u)}{u-1} + 3\,\mathrm{Li}_2(u) \end{aligned}$$

$$
-2\,\mathrm{Li}_3(1-u)-4\,\mathrm{Li}_3(u)-\frac{\pi^2}{6\,(u-1)}+\frac{13}{3}\,\zeta_3-\frac{17\pi^2}{24}-12\,, \tag{30}
$$

$$
\begin{aligned}
f_2(u) = & -\frac{1}{6}\ln^4(1-u)-\frac{2}{3}\ln^4(u)+\frac{4}{3}\ln^3(u)\ln(1-u)+\frac{2\ln^3(u)}{3\,(u-1)}+2\ln^3(u) \\
& -\frac{\pi^2}{3}\ln^2(1-u)-\ln^2(u)\ln^2(1-u)-\frac{\ln(1-u)\ln^2(u)}{u-1}-3\ln(1-u)\ln^2(u) \\
& -\frac{4\ln^2(u)}{u-1}-\frac{5\pi^2}{6}\ln^2(u)-8\ln^2(u)+\frac{2}{3}\ln(u)\ln^3(1-u)+\frac{4\ln(1-u)\ln(u)}{u-1} \\
& +\frac{5\pi^2}{6}\ln(u)\ln(1-u)+8\ln(u)\ln(1-u)+\frac{5\pi^2\ln(u)}{12\,(u-1)}+\frac{8\ln(u)}{u-1}+\frac{5\pi^2}{4}\ln(u) \\
& +16\ln(u)+\frac{4\,\mathrm{Li}_2(u)}{u-1}+\frac{\pi^2}{6}\mathrm{Li}_2(u)+8\,\mathrm{Li}_2(u)-\frac{\mathrm{Li}_3(1-u)}{u-1}-3\,\mathrm{Li}_3(1-u) \\
& -\frac{2\,\mathrm{Li}_3(u)}{u-1}-6\,\mathrm{Li}_3(u)+2\,\mathrm{Li}_4(1-u)+4\,\mathrm{Li}_4(u)-4\,\mathrm{Li}_4\left(\frac{u}{u-1}\right)-\frac{2\pi^2}{3\,(u-1)} \\
& -\frac{14}{3}\ln(u)\,\zeta_3+\frac{2\zeta_3}{u-1}+\frac{41}{6}\,\zeta_3-\frac{5\pi^4}{32}-\frac{11\pi^2}{6}-24\,,
\end{aligned} \tag{31}
$$

$$
g_0(u) = \frac{2\,u\ln(u)}{(u-1)^2}-\frac{2}{u-1}\,, \tag{32}
$$

$$
g_1(u) = -\frac{2\,u\ln^2(u)}{(u-1)^2}+\frac{2\,u\ln(u)}{(u-1)^2}-\frac{2\,u\,\mathrm{Li}_2(1-u)}{(u-1)^2}-\frac{4}{u-1}\,, \tag{33}
$$

$$
\begin{aligned}
g_2(u) = & \frac{2\,u}{(u-1)^2}\Bigg[\ln(u)\ln(1-u)+\frac{2}{3}\ln^3(u)-\ln^2(u)\ln(1-u)-\ln^2(u)+\frac{5\,\pi^2}{12}\ln(u) \\
& +2\ln(u)+\mathrm{Li}_2(u)-\mathrm{Li}_3(1-u)-2\,\mathrm{Li}_3(u)+2\zeta_3-\frac{\pi^2}{6}\Bigg]-\frac{(\pi^2+48)}{6\,(u-1)}\,,
\end{aligned} \tag{34}
$$

$$
h_0(u) = -\frac{u(2u-1)\ln(u)}{(u-1)^2}+\frac{u}{u-1}\,, \tag{35}
$$

$$
h_1(u) = \frac{u(2u-1)\ln^2(u)}{(u-1)^2}+\frac{u(2u-1)\,\mathrm{Li}_2(1-u)}{(u-1)^2}-\frac{u(5u-4)\ln(u)}{(u-1)^2}+\frac{2u}{u-1}\,, \tag{36}
$$

$$
\begin{aligned}
h_2(u) = & \frac{u(2u-1)}{(u-1)^2}\Bigg[\ln^2(u)\ln(1-u)-\frac{2}{3}\ln^3(u)-\frac{5\pi^2}{12}\ln(u)+\mathrm{Li}_3(1-u)+2\,\mathrm{Li}_3(u) \\
& -2\zeta_3-\ln(u)\ln(1-u)+\ln^2(u)-2\ln(u)-\mathrm{Li}_2(u)\Bigg]+\frac{\pi^2\,u}{6\,(u-1)^2}
\end{aligned}
$$

$$+\frac{3u}{u-1}\left[\ln^2(u)-\ln(u)\ln(1-u)-\,\mathrm{Li}_2(u)-2\ln(u)+\frac{11\pi^2}{36}+\frac{4}{3}\right]\,. \tag{37}$$

4.2 Two-loop results

The two-loop part of the form factor F_1 reads

$$\begin{aligned}
F_1^{(2)} &= C_F^2\left\{\frac{1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left[L+j_{-3}(u)\right]+\frac{1}{\epsilon^2}\left[L^2+2L\,j_{-3}(u)+j_{-2}(u)\right]\right.\\
&\quad+\frac{1}{\epsilon}\left[\frac{2L^3}{3}+2L^2\,j_{-3}(u)+2L\,j_{-2}(u)+j_{-1}(u)\right]\\
&\quad\left.+\left[\frac{L^4}{3}+\frac{4}{3}L^3\,j_{-3}(u)+2L^2\,j_{-2}(u)+2L\,j_{-1}(u)+j_0(u)\right]\right\}\\
&\quad+C_F\,C_A\left\{-\frac{11}{12\epsilon^3}+\frac{1}{\epsilon^2}\left[-\frac{11}{6}L+k_{-2}(u)\right]+\frac{1}{\epsilon}\left[-\frac{11}{6}L^2+2L\,k_{-2}(u)+k_{-1}(u)\right]\right.\\
&\quad\left.+\left[-\frac{11}{9}L^3+2L^2\,k_{-2}(u)+2L\,k_{-1}(u)+k_0(u)\right]\right\}\\
&\quad+C_F\,t_f\,n_f\left\{\frac{1}{3\epsilon^3}+\frac{1}{\epsilon^2}\left[\frac{2}{3}L+p_{-2}(u)\right]+\frac{1}{\epsilon}\left[\frac{2}{3}L^2+2L\,p_{-2}(u)+p_{-1}(u)\right]\right.\\
&\quad\left.+\left[\frac{4}{9}L^3+2L^2\,p_{-2}(u)+2L\,p_{-1}(u)+p_0(u)\right]\right\}\\
&\quad+C_F\,t_f\left\{\frac{4}{3\epsilon^3}+\frac{1}{\epsilon^2}\left[\frac{8}{3}L+q_{-2}(u)\right]+\frac{1}{\epsilon}\left[\frac{8}{3}L^2+2L\,q_{-2}(u)+q_{-1}(u)\right]\right.\\
&\quad\left.+\left[\frac{16}{9}L^3+2L^2\,q_{-2}(u)+2L\,q_{-1}(u)+q_0(u)\right]\right\}
\end{aligned} \tag{38}$$

with

$$j_{-3}(u) = \frac{5}{2}-2\ln(u)\,, \tag{39}$$

$$j_{-2}(u) = 4\ln^2(u)-2\ln(u)\ln(1-u)-2\,\mathrm{Li}_2(u)-\frac{\ln(u)}{u-1}-8\ln(u)+\frac{5\pi^2}{12}+\frac{73}{8}\,, \tag{40}$$

$$\begin{aligned}
j_{-1}(u) &= -\frac{16}{3}\ln^3(u)+6\ln^2(u)\ln(1-u)+\frac{3\ln^2(u)}{u-1}+14\ln^2(u)-\frac{\ln(1-u)\ln(u)}{u-1}\\
&\quad-8\ln(u)\ln(1-u)-\frac{13\ln(u)}{2\,(u-1)}-\frac{5\pi^2}{3}\ln(u)-\frac{55}{2}\ln(u)+4\ln(u)\,\mathrm{Li}_2(u)-\frac{\mathrm{Li}_2(u)}{u-1}\\
&\quad-8\,\mathrm{Li}_2(u)+2\,\mathrm{Li}_3(1-u)+4\,\mathrm{Li}_3(u)+\frac{\pi^2}{6\,(u-1)}-\frac{31}{3}\zeta_3+\frac{9\pi^2}{4}+\frac{213}{8}\,,
\end{aligned} \tag{41}$$

$$
\begin{aligned}
j_0(u) = {} & \frac{(3u^3 - 8u^2 + 52u - 56)}{6\,u^3}\Bigg[\ln^4(1-u) + 2\pi^2\ln^2(1-u) - 4\ln^3(1-u)\ln(u) \\
& + 24\,\mathrm{Li}_4\!\left(\frac{u}{u-1}\right)\Bigg] + \frac{8\,(5u^3 + 12u^2 - 78u + 84)}{3\,u^3}\ln(u)\,\zeta_3 - \frac{28}{3}\ln^3(u)\ln(1-u) \\
& + \frac{(13u^3 - 12u^2 + 78u - 84)}{u^3}\ln^2(u)\ln^2(1-u) + \frac{5\ln(1-u)\ln^2(u)}{u-1} \\
& + \frac{(30u^3 - 39u^2 + 56u - 3)}{u^3}\ln^2(u)\ln(1-u) + \frac{10\pi^2}{3}\ln^2(u) - \frac{15\ln(1-u)\ln(u)}{2\,(u-1)} \\
& + \frac{2\,(5u^3 - 2u^2 + 13u - 14)}{u^3}\left[\mathrm{Li}_2^2(u) + 2\ln(u)\ln(1-u)\,\mathrm{Li}_2(u)\right] - \frac{14\ln^3(u)}{3\,(u-1)} \\
& + \frac{(u^3 + 2u - 2)}{u^3}\left[-16\,\mathrm{HPL}(\{-2,2\},1-u) + \frac{4\pi^2}{3}\mathrm{Li}_2(u-1)\right] + \frac{25\ln^2(u)}{u} \\
& + \frac{8\,(5u^2 - 10u + 4)}{u^2}\left[\mathrm{HPL}(\{-1,2\},1-u) + \frac{\pi^2}{12}\ln(2-u)\right] + \frac{59}{2}\ln^2(u) \\
& - \frac{8}{u-1}\,\mathrm{HPL}(\{-1,2\},1-u) - \frac{2\pi^2\ln(2-u)}{3\,(u-1)} + \frac{16}{3}\ln^4(u) - \frac{52}{3}\ln^3(u) \\
& - \frac{\pi^2\,(17u^3 - 32u^2 + 192u - 208)}{3\,u^3}\ln(1-u)\ln(u) - \frac{153}{2}\ln(u) + \frac{25\ln^2(u)}{2\,(u-1)} \\
& + \frac{(19u^2 - 180u + 100)}{2\,u^2}\left[\ln(1-u)\ln(u) + \mathrm{Li}_2(u)\right] + \frac{\pi^2\ln(u)}{6\,(u-1)} - \frac{49\ln(u)}{2\,(u-1)} \\
& + \frac{(u^2-3)}{u^3}\left[\ln^2(u)\ln(1+u) + \pi^2\ln(1+u) + 2\ln(u)\,\mathrm{Li}_2(-u) - 2\,\mathrm{Li}_3(-u)\right] \\
& - \frac{8\pi^2(u+3)(3u-4)}{3\,u^2}\ln(u) + \frac{2\,(16u^3 - 51u^2 + 84u - 3)}{u^3}\ln(u)\,\mathrm{Li}_2(u) \\
& - \frac{15\,\mathrm{Li}_2(u)}{2\,(u-1)} + \frac{6\ln(u)\,\mathrm{Li}_2(u)}{u-1} - \frac{\pi^2\,(11u^3 - 16u^2 + 88u - 96)}{3\,u^3}\mathrm{Li}_2(u) \\
& + \frac{4\,(5u^3 - 8u^2 + 52u - 56)}{u^3}\ln(u)\,\mathrm{Li}_3(1-u) - 4\ln^2(u)\,\mathrm{Li}_2(u) - \frac{\mathrm{Li}_3(1-u)}{u-1} \\
& - \frac{6\,(2u^3 - 2u^2 - 4u - 1)}{u^3}\mathrm{Li}_3(1-u) - 8\ln(u)\,\mathrm{Li}_3(u) - \frac{2\,\mathrm{Li}_3(u)}{u-1} \\
& - \frac{2\,(2u^3 - 63u^2 + 112u - 3)}{u^3}\mathrm{Li}_3(u) - \frac{2\,(9u^3 - 20u^2 + 114u - 124)}{u^3}\mathrm{Li}_4(1-u) \\
& + \frac{4\,(u^3 - 8u^2 + 52u - 56)}{u^3}\mathrm{Li}_4(u) + \frac{\pi^2\,(21u^2 + 1168u - 1344)}{48\,u^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\pi^4\,(2453u^3-960u^2+6096u-6576)}{2160\,u^3}+\frac{2\,(14u^3-234u^2+336u-9)\zeta_3}{3\,u^3}\\
&-\frac{4\,\pi^2\,(u-4)\ln(2)}{u}+\frac{5\pi^2}{4\,(u-1)}+\frac{2\zeta_3}{u-1}+\frac{1327}{16}+2\,c_0\,,
\end{aligned}
\tag{42}
$$

$$
k_{-2}(u) = \frac{11}{3}\ln(u)+\frac{\pi^2}{12}-\frac{58}{9}\,,
\tag{43}
$$

$$
\begin{aligned}
k_{-1}(u) = &-\frac{22}{3}\ln^2(u)+\frac{22}{3}\ln(u)\ln(1-u)+\frac{22}{3}\,\mathrm{Li}_2(u)+\frac{11\ln(u)}{3\,(u-1)}-\frac{\pi^2}{3}\ln(u)\\
&+\frac{166}{9}\ln(u)+\frac{11}{2}\zeta_3-\frac{131\pi^2}{72}-\frac{6301}{216}\,,
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
k_0(u) = &-\frac{(4u^2-10u+7)}{3\,u^3}\Bigg[\ln^4(1-u)+2\pi^2\ln^2(1-u)-4\ln^3(1-u)\ln(u)\\
&+24\,\mathrm{Li}_4(u)+24\,\mathrm{Li}_4\!\left(\frac{u}{u-1}\right)\Bigg]-\frac{2\,(7u^3-16u^2+40u-28)}{u^3}\ln(u)\,\zeta_3\\
&-\frac{(4u^3+12u^2-30u+21)}{u^3}\ln^2(u)\ln^2(1-u)\\
&-\frac{(106u^3+33u^2-84u-9)}{6\,u^3}\ln^2(u)\ln(1-u)+\frac{2\pi^2}{3}\ln^2(u)+\frac{22\ln(1-u)\ln(u)}{3\,(u-1)}\\
&-\frac{(4u^3+4u^2-10u+7)}{u^3}\left[\mathrm{Li}_2^2(u)+2\ln(u)\ln(1-u)\,\mathrm{Li}_2(u)\right]\\
&+\frac{(u^3+2u-2)}{u^3}\left[8\,\mathrm{HPL}(\{-2,2\},1-u)-\frac{2\pi^2}{3}\,\mathrm{Li}_2(u-1)\right]+\frac{17\ln^2(u)}{2\,u}\\
&-\frac{4\,(5u^2-10u+4)}{u^2}\left[\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2}{12}\ln(2-u)\right]-\frac{547}{18}\ln^2(u)\\
&+\frac{4}{u-1}\,\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2\ln(2-u)}{3\,(u-1)}+\frac{88}{9}\ln^3(u)\\
&+\frac{8\pi^2\,(4u^2-11u+8)}{3\,u^3}\ln(1-u)\ln(u)+\frac{4129}{54}\ln(u)-\frac{35\ln^2(u)}{6\,(u-1)}\\
&+\frac{(215u^2-108u+153)}{9\,u^2}\left[\ln(1-u)\ln(u)+\mathrm{Li}_2(u)\right]-\frac{2\pi^2\ln(u)}{3\,(u-1)}+\frac{533\ln(u)}{18\,(u-1)}\\
&-\frac{(u^2-3)}{2\,u^3}\left[\ln^2(u)\ln(1+u)+\pi^2\ln(1+u)+2\ln(u)\,\mathrm{Li}_2(-u)-2\,\mathrm{Li}_3(-u)\right]\\
&+\frac{\pi^2(155u^2-180u+216)}{18\,u^2}\ln(u)-\frac{3\,(2u^3+8u^2-14u-1)}{u^3}\ln(u)\,\mathrm{Li}_2(u)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{22\,\mathrm{Li}_2(u)}{3\,(u-1)}+\frac{4\pi^2\,(2u-3)^2}{3\,u^3}\,\mathrm{Li}_2(u)-\frac{8\,(u^3+4u^2-10u+7)}{u^3}\ln(u)\,\mathrm{Li}_3(1-u)\\
&-\frac{(2u^3+93u^2-90u+9)}{3\,u^3}\,\mathrm{Li}_3(1-u)-\frac{(70u^3-111u^2+168u+9)}{3\,u^3}\,\mathrm{Li}_3(u)\\
&+\frac{2\,(20u^2-58u+43)}{u^3}\,\mathrm{Li}_4(1-u)-\frac{\pi^2\,(230u^2-405u+378)}{27\,u^2}\\
&-\frac{\pi^4\,(269u^3+480u^2-1236u+876)}{1080\,u^3}+\frac{(461u^3-396u^2+1008u+54)\zeta_3}{18\,u^3}\\
&+\frac{2\,\pi^2\,(u-4)\ln(2)}{u}-\frac{11\pi^2}{9\,(u-1)}-\frac{146461}{1296}-c_0\;, \qquad (45)
\end{aligned}
$$

$$
p_{-2}(u)=\frac{20}{9}-\frac{4}{3}\ln(u)\;, \qquad (46)
$$

$$
p_{-1}(u)=\frac{8}{3}\ln^2(u)-\frac{8}{3}\ln(u)\ln(1-u)-\frac{8}{3}\,\mathrm{Li}_2(u)-\frac{4\ln(u)}{3\,(u-1)}-\frac{56}{9}\ln(u)+\frac{13\pi^2}{18}+\frac{557}{54}\;, \qquad (47)
$$

$$
\begin{aligned}
p_0(u)=&-\frac{32}{9}\ln^3(u)+\frac{16}{3}\ln^2(u)\ln(1-u)+\frac{8\ln^2(u)}{3\,(u-1)}+\frac{112\ln^2(u)}{9}-\frac{8\ln(u)\ln(1-u)}{3\,(u-1)}\\
&-\frac{112}{9}\ln(u)\ln(1-u)-\frac{86\ln(u)}{9\,(u-1)}-\frac{26\pi^2}{9}\ln(u)-\frac{706}{27}\ln(u)-\frac{8\,\mathrm{Li}_2(u)}{3\,(u-1)}-\frac{74}{9}\zeta_3\\
&-\frac{112}{9}\,\mathrm{Li}_2(u)+\frac{16}{3}\,\mathrm{Li}_3(1-u)+\frac{32}{3}\,\mathrm{Li}_3(u)+\frac{4\pi^2}{9\,(u-1)}+\frac{106\pi^2}{27}+\frac{11813}{324}\;, \qquad (48)
\end{aligned}
$$

$$
q_{-2}(u)=\frac{10}{3}-\frac{8}{3}\ln(u)\;, \qquad (49)
$$

$$
q_{-1}(u)=\frac{8}{3}\ln^2(u)-\frac{8}{3}\ln(u)\ln(1-u)-\frac{8}{3}\,\mathrm{Li}_2(u)-\frac{4\ln(u)}{3\,(u-1)}-4\ln(u)+\frac{2\pi^2}{3}+8\;, \qquad (50)
$$

$$
\begin{aligned}
q_0(u)=&-\frac{16}{9}\ln^3(u)+\frac{8}{3}\ln^2(u)\ln(1-u)+\frac{4\ln^2(u)}{3\,(u-1)}+4\ln^2(u)-\frac{8\ln(u)\ln(1-u)}{3\,(u-1)}\\
&+\frac{8\,(5u^3-39u^2+54u-16)}{9\,u^3}\left[\ln(u)\ln(1-u)+\mathrm{Li}_2(u)\right]-\frac{86\ln(u)}{9\,(u-1)}-\frac{4\pi^2}{3}\ln(u)\\
&-\frac{2\,(409u^2-660u+192)}{27\,u^2}\ln(u)-\frac{8\,\mathrm{Li}_2(u)}{3\,(u-1)}+\frac{16\,(u^3-3u+3)}{3\,u^3}\,\mathrm{Li}_3(1-u)\\
&+\frac{16}{3}\,\mathrm{Li}_3(u)+\frac{4\pi^2}{9\,(u-1)}-\frac{\pi^2(11u^2+12u-24)}{9\,u^2}-\frac{16\,(5u^3-9u+9)}{9\,u^3}\zeta_3
\end{aligned}
$$

$$
+\frac{10543u^2 - 9144u + 2304}{162\,u^2} \,. \tag{51}
$$

We comment on the rôle of the constant c_0 in the Appendix. The two-loop part of the form factor F_2 reads

$$
\begin{aligned}
F_2^{(2)} = {} & C_F^2 \left\{ \frac{1}{\epsilon^2}\, r_{-2}(u) + \frac{1}{\epsilon}\left[2L\, r_{-2}(u) + r_{-1}(u)\right] + \left[2L^2\, r_{-2}(u) + 2L\, r_{-1}(u) + r_0(u)\right] \right\} \\
& + C_F\, C_A \left\{ \frac{1}{\epsilon}\, s_{-1}(u) + \left[2L\, s_{-1}(u) + s_0(u)\right] \right\} \\
& + C_F\, t_f\, n_f \left\{ \frac{1}{\epsilon}\, t_{-1}(u) + \left[2L\, t_{-1}(u) + t_0(u)\right] \right\} \\
& + C_F\, t_f \left\{ \frac{1}{\epsilon}\, v_{-1}(u) + \left[2L\, v_{-1}(u) + v_0(u)\right] \right\}
\end{aligned} \tag{52}
$$

with

$$
r_{-2}(u) = -g_0(u) \,, \tag{53}
$$

$$
r_{-1}(u) = \frac{2\,u}{(u-1)^2}\left[\,\mathrm{Li}_2(1-u) + 3\ln^2(u)\right] - \frac{(11u-4)\ln(u)}{(u-1)^2} + \frac{9}{u-1} \,, \tag{54}
$$

$$
\begin{aligned}
r_0(u) = {} & \frac{8\,(2u-7)}{3\,u^3}\Bigg[\ln^4(1-u) + 2\pi^2\ln^2(1-u) - 4\ln^3(1-u)\ln(u) \\
& -24\ln(u)\,\zeta_3 + 9\ln^2(u)\ln^2(1-u) + 3\,\mathrm{Li}_2^2(u) + 6\ln(u)\ln(1-u)\,\mathrm{Li}_2(u) \\
& +24\ln(u)\,\mathrm{Li}_3(1-u) + 24\,\mathrm{Li}_4(u) + 24\,\mathrm{Li}_4\!\left(\frac{u}{u-1}\right)\Bigg] + \frac{50\ln^2(u)}{u} \\
& + \frac{2\,(11u^2+52u-3)}{u^3}\ln^2(u)\ln(1-u) + \frac{(23u-36)}{(u-1)^2}\left[\ln(1-u)\ln(u) + \mathrm{Li}_2(u)\right] \\
& + \frac{8}{u^3}\left[8\,\mathrm{HPL}(\{-2,2\},1-u) - \frac{2\pi^2}{3}\,\mathrm{Li}_2(u-1)\right] - \frac{2(3u-8)}{(u-1)^2}\ln^2(u)\ln(1-u) \\
& + \frac{32\,(u+2)}{u^2}\left[\mathrm{HPL}(\{-1,2\},1-u) + \frac{\pi^2}{12}\ln(2-u)\right] - \frac{(25u^2-49u+28)}{(u-1)^3}\ln^2(u) \\
& - \frac{16\,u}{(u-1)^2}\left[\mathrm{HPL}(\{-1,2\},1-u) + \frac{\pi^2}{12}\ln(2-u) + \frac{7}{12}\ln^3(u)\right] \\
& - \frac{32\pi^2\,(4u-13)}{3\,u^3}\ln(1-u)\ln(u) - \frac{4\,(11u-25)}{u^2}\left[\ln(1-u)\ln(u) + \mathrm{Li}_2(u)\right] \\
& - \frac{2\,(u^2+4u+3)}{u^3}\left[\ln^2(u)\ln(1+u) + \pi^2\ln(1+u) + 2\ln(u)\,\mathrm{Li}_2(-u) - 2\mathrm{Li}_3(-u)\right]
\end{aligned}
$$

$$+\frac{32\pi^2(u+6)}{3\,u^2}\ln(u)+\frac{4(17u^2+80u-3)}{u^3}\ln(u)\,\mathrm{Li}_2(u)-\frac{12(3u-4)}{(u-1)^2}\ln(u)\,\mathrm{Li}_2(u)$$

$$-\frac{64\pi^2\,(u-3)}{3\,u^3}\mathrm{Li}_2(u)+\frac{4\,(8u+3)}{u^3}\mathrm{Li}_3(1-u)-\frac{2\,(9u-8)}{(u-1)^2}\mathrm{Li}_3(1-u)$$

$$-\frac{4\,(23u^2+108u-3)}{u^3}\mathrm{Li}_3(u)+\frac{4\,(15u-16)}{(u-1)^2}\mathrm{Li}_3(u)-\frac{16\,(10u-31)}{u^3}\mathrm{Li}_4(1-u)$$

$$+\frac{2\pi^2\,(25u-84)}{3\,u^2}-\frac{\pi^2\,(41u-54)}{6\,(u-1)^2}+\frac{2\pi^4\,(40u-137)}{45\,u^3}-\frac{\pi^2(31u-32)}{3\,(u-1)^2}\ln(u)$$

$$-\frac{8\,(3u-1)}{(u-1)^2}\ln(u)+\frac{4\,(22u^2+116u-3)\zeta_3}{u^3}-\frac{4\,(15u-16)}{(u-1)^2}\zeta_3+\frac{31}{u-1}\;, \tag{55}$$

$$s_{-1}(u)=\frac{22u\ln(u)}{3\,(u-1)^2}-\frac{22}{3\,(u-1)}\;, \tag{56}$$

$$s_0(u)=\frac{2\,(4u-7)}{3\,u^3}\Bigg[\ln^4(1-u)+2\pi^2\ln^2(1-u)-4\ln^3(1-u)\ln(u)$$

$$-24\ln(u)\,\zeta_3+9\ln^2(u)\ln^2(1-u)+3\,\mathrm{Li}_2^2(u)+6\ln(u)\ln(1-u)\,\mathrm{Li}_2(u)$$

$$+24\ln(u)\,\mathrm{Li}_3(1-u)+24\,\mathrm{Li}_4(u)+24\,\mathrm{Li}_4\!\left(\frac{u}{u-1}\right)\Bigg]+\frac{17\ln^2(u)}{u}$$

$$-\frac{(u^2-32u-3)}{u^3}\ln^2(u)\ln(1-u)+\frac{4\,(17u-6)}{3\,(u-1)^2}\left[\ln(1-u)\ln(u)+\,\mathrm{Li}_2(u)\right]$$

$$-\frac{4}{u^3}\left[8\,\mathrm{HPL}(\{-2,2\},1-u)-\frac{2\pi^2}{3}\mathrm{Li}_2(u-1)\right]-\frac{4}{u-1}\ln^2(u)\ln(1-u)$$

$$-\frac{16\,(u+2)}{u^2}\left[\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2}{12}\ln(2-u)\right]-\frac{5\,(13u-6)}{3\,(u-1)^2}\ln^2(u)$$

$$+\frac{8\,u}{(u-1)^2}\left[\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2}{12}\ln(2-u)\right]$$

$$-\frac{64\pi^2\,(u-2)}{3\,u^3}\ln(1-u)\ln(u)-\frac{2\,(16u-17)}{u^2}\left[\ln(1-u)\ln(u)+\,\mathrm{Li}_2(u)\right]$$

$$+\frac{(u^2+4u+3)}{u^3}\left[\ln^2(u)\ln(1+u)+\pi^2\ln(1+u)+2\ln(u)\,\mathrm{Li}_2(-u)-2\mathrm{Li}_3(-u)\right]$$

$$+\frac{4\pi^2(u+18)}{3\,u^2}\ln(u)-\frac{2(2u^2-46u-3)}{u^3}\ln(u)\,\mathrm{Li}_2(u)-\frac{12}{u-1}\ln(u)\,\mathrm{Li}_2(u)$$

$$-\frac{8\pi^2\,(4u-9)}{3\,u^3}\mathrm{Li}_2(u)+\frac{2\,(5u^2+34u-3)}{u^3}\mathrm{Li}_3(1-u)-\frac{4\,\mathrm{Li}_3(1-u)}{u-1}$$

$$+\frac{6\,(u^2-20u-1)}{u^3}\,\mathrm{Li}_3(u)+\frac{16\,\mathrm{Li}_3(u)}{u-1}-\frac{4\,(20u-43)}{u^3}\,\mathrm{Li}_4(1-u)$$

$$+\frac{2\pi^2\,(3u-14)}{u^2}-\frac{2\pi^2\,(8u+3)}{9\,(u-1)^2}+\frac{\pi^4\,(40u-73)}{45\,u^3}-\frac{4\pi^2(3u-2)}{3\,(u-1)^2}\ln(u)$$

$$+\frac{(323u-54)}{9\,(u-1)^2}\ln(u)-\frac{2\,(2u^2-52u-3)\zeta_3}{u^3}-\frac{16\,\zeta_3}{u-1}-\frac{401}{9\,(u-1)}\,, \tag{57}$$

$$t_{-1}(u)=-\frac{8u\ln(u)}{3\,(u-1)^2}+\frac{8}{3\,(u-1)}\,, \tag{58}$$

$$t_0(u)=\frac{4\,u}{9\,(u-1)^2}\left[12\ln^2(u)+12\,\mathrm{Li}_2(1-u)-19\ln(u)\right]+\frac{124}{9\,(u-1)}\,, \tag{59}$$

$$v_{-1}(u)=t_{-1}(u)\,, \tag{60}$$

$$v_0(u)=\frac{8\,u}{3\,(u-1)^2}\left[\ln^2(u)+2\,\mathrm{Li}_2(1-u)\right]-\frac{4(79u-60)}{9\,(u-1)^2}\ln(u)+\frac{80\ln(u)}{3\,u}$$

$$+\frac{8\,(u+2)}{3\,u^2}\left[2\ln(u)\ln(1-u)+2\,\mathrm{Li}_2(u)+\pi^2\right]+\frac{124}{9\,(u-1)}$$

$$+\frac{32\,\mathrm{Li}_3(1-u)}{u^3}-\frac{32\zeta_3}{u^3}-\frac{104}{3\,u}\,. \tag{61}$$

Finally, the two-loop part of the form factor F_3 reads

$$F_3^{(2)}=C_F^2\left\{\frac{1}{\epsilon^2}\,w_{-2}(u)+\frac{1}{\epsilon}\left[2L\,w_{-2}(u)+w_{-1}(u)\right]+\left[2L^2\,w_{-2}(u)+2L\,w_{-1}(u)+w_0(u)\right]\right\}$$

$$+C_F\,C_A\left\{\frac{1}{\epsilon}\,x_{-1}(u)+\left[2L\,x_{-1}(u)+x_0(u)\right]\right\}$$

$$+C_F\,t_f\,n_f\left\{\frac{1}{\epsilon}\,y_{-1}(u)+\left[2L\,y_{-1}(u)+y_0(u)\right]\right\}$$

$$+C_F\,t_f\left\{\frac{1}{\epsilon}\,z_{-1}(u)+\left[2L\,z_{-1}(u)+z_0(u)\right]\right\} \tag{62}$$

with

$$w_{-2}(u)=-h_0(u)\,, \tag{63}$$

$$w_{-1}(u)=-\frac{u(2u-1)}{(u-1)^2}\left[3\ln^2(u)+\,\mathrm{Li}_2(1-u)\right]+\frac{u(24u-17)}{2\,(u-1)^2}\ln(u)-\frac{9u}{2\,(u-1)}\,, \tag{64}$$

$$w_0(u)=\frac{4\,(2u^2-16u+21)}{3\,u^3}\Bigg[\ln^4(1-u)+2\pi^2\ln^2(1-u)-4\ln^3(1-u)\ln(u)$$

$$-24\ln(u)\,\zeta_3+9\ln^2(u)\ln^2(1-u)+3\,\mathrm{Li}_2^2(u)+6\ln(u)\ln(1-u)\,\mathrm{Li}_2(u)$$

$$
\begin{aligned}
&+24\ln(u)\,\mathrm{Li}_3(1-u)+24\,\mathrm{Li}_4(u)+24\,\mathrm{Li}_4\left(\frac{u}{u-1}\right)\bigg]-\frac{75\ln^2(u)}{u}-\frac{31u}{2\,(u-1)}\\
&-\frac{(16u^3-47u^2+160u-9)}{u^3}\ln^2(u)\ln(1-u)+\frac{6(u-2)}{(u-1)^2}\ln(u)\,\mathrm{Li}_2(u)\\
&+\frac{(u+12)}{2\,(u-1)^2}\left[\ln(1-u)\ln(u)+\mathrm{Li}_2(u)\right]-\frac{(7u-2)}{(u-1)^2}\ln^2(u)\ln(1-u)+\frac{28\ln^3(u)}{3}\\
&+\frac{32(2u-3)}{u^3}\left[\mathrm{HPL}(\{-2,2\},1-u)-\frac{\pi^2}{12}\mathrm{Li}_2(u-1)\right]-\frac{2\,(13u-14)}{(u-1)^2}\mathrm{Li}_3(u)\\
&-\frac{16\,(u^2-u+6)}{u^2}\left[\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2}{12}\ln(2-u)\right]+\frac{\pi^2(29u-30)}{6\,(u-1)^2}\ln(u)\\
&+\frac{8\,(3u-2)}{(u-1)^2}\left[\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2}{12}\ln(2-u)+\frac{7}{12}\ln^3(u)\right]+8\pi^2\ln(2)\\
&-\frac{(24u^3-73u^2+69u-24)}{2\,(u-1)^3}\ln^2(u)-\frac{16\pi^2\,(4u^2-30u+39)}{3\,u^3}\ln(1-u)\ln(u)\\
&-\frac{2\,(6u^2-71u+75)}{u^2}\left[\ln(1-u)\ln(u)+\mathrm{Li}_2(u)\right]-\frac{\pi^2(u^2-80u+288)}{3\,u^2}\ln(u)\\
&+\frac{(2u^3+3u^2+8u+9)}{u^3}\big[\ln^2(u)\ln(1+u)+\pi^2\ln(1+u)+2\ln(u)\,\mathrm{Li}_2(-u)\\
&-2\mathrm{Li}_3(-u)\big]-\frac{2(10u^3-69u^2+244u-9)}{u^3}\ln(u)\,\mathrm{Li}_2(u)+\frac{(11u-10)}{(u-1)^2}\mathrm{Li}_3(1-u)\\
&-\frac{32\pi^2\,(u^2-7u+9)}{3\,u^3}\mathrm{Li}_2(u)+\frac{2\,(3u^3+8u^2-28u-9)}{u^3}\mathrm{Li}_3(1-u)\\
&+\frac{2\,(4u^3-91u^2+328u-9)}{u^3}\mathrm{Li}_3(u)-\frac{8\,(10u^2-72u+93)}{u^3}\mathrm{Li}_4(1-u)\\
&+\frac{\pi^2\,(5u^2-390u+504)}{6\,u^2}+\frac{\pi^2\,(17u-30)}{12\,(u-1)^2}-\frac{2\,(14u^3-94u^2+344u-9)\zeta_3}{u^3}\\
&+\frac{(79u^2-67u+4)}{2\,(u-1)^2}\ln(u)+\frac{\pi^4\,(40u^2-314u+411)}{45\,u^3}+\frac{2\,(13u-14)}{(u-1)^2}\zeta_3\,, \qquad (65)
\end{aligned}
$$

$$
x_{-1}(u)=-\frac{11u(2u-1)}{3\,(u-1)^2}\ln(u)+\frac{11u}{3\,(u-1)}\,, \qquad (66)
$$

$$
\begin{aligned}
x_0(u)=\;&\frac{(8u^2-24u+21)}{3\,u^3}\bigg[\ln^4(1-u)+2\pi^2\ln^2(1-u)-4\ln^3(1-u)\ln(u)\\
&-24\ln(u)\,\zeta_3+9\ln^2(u)\ln^2(1-u)+3\,\mathrm{Li}_2^2(u)+6\ln(u)\ln(1-u)\,\mathrm{Li}_2(u)
\end{aligned}
$$

$$
\begin{aligned}
&+24\ln(u)\,\mathrm{Li}_3(1-u)+24\,\mathrm{Li}_4(u)+24\,\mathrm{Li}_4\left(\frac{u}{u-1}\right)\bigg]-\frac{51\ln^2(u)}{2\,u}+\frac{347u+54}{18\,(u-1)}\\
&-\frac{(2u^3-51u^2+92u+9)}{2\,u^3}\ln^2(u)\ln(1-u)+\frac{6}{u-1}\ln(u)\,\mathrm{Li}_2(u)\\
&-\frac{2(39u-28)}{3\,(u-1)^2}\left[\ln(1-u)\ln(u)+\,\mathrm{Li}_2(u)\right]+\frac{2}{u-1}\ln^2(u)\ln(1-u)\\
&-\frac{16(2u-3)}{u^3}\left[\mathrm{HPL}(\{-2,2\},1-u)-\frac{\pi^2}{12}\,\mathrm{Li}_2(u-1)\right]-\frac{8}{u-1}\,\mathrm{Li}_3(u)\\
&+\frac{8\,(u^2-u+6)}{u^2}\left[\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2}{12}\ln(2-u)\right]+\frac{2\pi^2(5u-4)}{3\,(u-1)^2}\ln(u)\\
&-\frac{4\,(3u-2)}{(u-1)^2}\left[\mathrm{HPL}(\{-1,2\},1-u)+\frac{\pi^2}{12}\ln(2-u)\right]-4\pi^2\ln(2)\\
&+\frac{(94u^2-53u-6)}{6\,(u-1)^2}\ln^2(u)-\frac{16\pi^2\,(4u^2-13u+12)}{3\,u^3}\ln(1-u)\ln(u)\\
&-\frac{(56u^2-216u+153)}{3\,u^2}\left[\ln(1-u)\ln(u)+\,\mathrm{Li}_2(u)\right]-\frac{2\pi^2(2u^2-29u+54)}{3\,u^2}\ln(u)\\
&-\frac{(2u^3+3u^2+8u+9)}{2\,u^3}\big[\ln^2(u)\ln(1+u)+\pi^2\ln(1+u)+2\ln(u)\,\mathrm{Li}_2(-u)\\
&-2\mathrm{Li}_3(-u)\big]-\frac{(8u^3-78u^2+134u+9)}{u^3}\ln(u)\,\mathrm{Li}_2(u)+\frac{2\,\mathrm{Li}_3(1-u)}{u-1}\\
&-\frac{4\pi^2\,(8u^2-28u+27)}{3\,u^3}\,\mathrm{Li}_2(u)-\frac{(6u^3-41u^2+98u-9)}{u^3}\,\mathrm{Li}_3(1-u)\\
&+\frac{(14u^3-105u^2+176u+9)}{u^3}\,\mathrm{Li}_3(u)-\frac{2\,(40u^2-136u+129)}{u^3}\,\mathrm{Li}_4(1-u)\\
&+\frac{\pi^2\,(76u^2-285u+378)}{9\,u^2}+\frac{\pi^2\,(30u-19)}{9\,(u-1)^2}-\frac{(4u^3-102u^2+160u+9)\zeta_3}{u^3}\\
&-\frac{(760u^2-329u-162)}{18\,(u-1)^2}\ln(u)+\frac{\pi^4\,(80u^2-246u+219)}{90\,u^3}+\frac{8\zeta_3}{u-1}\,, \qquad (67)
\end{aligned}
$$

$$
y_{-1}(u)=\frac{4u(2u-1)}{3\,(u-1)^2}\ln(u)-\frac{4u}{3\,(u-1)}\,, \qquad (68)
$$

$$
y_0(u)=-\frac{8\,u(2u-1)}{3\,(u-1)^2}\left[\ln^2(u)+\,\mathrm{Li}_2(1-u)\right]+\frac{2\,u(68u-49)}{9\,(u-1)^2}\ln(u)-\frac{62u}{9\,(u-1)}\,, \qquad (69)
$$

$$
z_{-1}(u)=y_{-1}(u)\,, \qquad (70)
$$

$$
\begin{aligned}
z_0(u) = & -\frac{4\,u(2u-1)}{3\,(u-1)^2}\ln^2(u) - \frac{8\,(3u-2)}{3\,(u-1)^2}\,\mathrm{Li}_2(1-u) + \frac{2(147u-128)}{9\,(u-1)^2}\ln(u) \\
& + \frac{8(17u-93)}{9\,u}\ln(u) + \frac{8\,(9u-22)}{3\,u^2}\left[\ln(u)\ln(1-u) + \mathrm{Li}_2(u)\right] - \frac{16(2u-3)}{u^3}\zeta_3 \\
& - \frac{2(55u-24)}{9\,(u-1)} - \frac{4\pi^2(2u^2-3u+18)}{9\,u^2} + \frac{16(2u-3)}{u^3}\,\mathrm{Li}_3(1-u) + \frac{284}{3\,u}\,. \qquad (71)
\end{aligned}
$$

As already mentioned in the introduction, our results have been compared analytically with those of [10] and complete agreement has been obtained.

5 Numerical evaluation and conclusion

In Fig. 3 we show the coefficient functions $C_i(u)$ in the one- (dashed) and two-loop (solid) approximation. For this purpose, we define the renormalized matching coefficients in the $\overline{\mathrm{MS}}$ scheme by subtracting minimally the infrared poles. More precisely, from $C_i = Z_J^{-1} F_i$, we obtain

$$
C_i = C_i^{(0)} + \frac{\alpha_s^{(4)}}{4\pi}\,C_i^{(1)} + \frac{{\alpha_s^{(4)}}^2}{(4\pi)^2}\,C_i^{(2)} + \mathcal{O}(\alpha_s^3)\,, \qquad (72)
$$

with

$$
\begin{aligned}
C_i^{(0)} &= F_i^{(0)}, & (73)\\
C_i^{(1)} &= F_i^{(1)} + [Z_J^{-1}]^{(1)}\,F_i^{(0)}, & (74)\\
C_i^{(2)} &= F_i^{(2)} + F_i^{(1)}\left\{[Z_J^{-1}]^{(1)} + \xi_{45}^{(1)} + Z_\alpha^{(1)}\right\} + [Z_J^{-1}]^{(2)}\,F_i^{(0)}\,. & (75)
\end{aligned}
$$

Expressions for the $F_i^{(j)}$ have been given in earlier sections. $[Z_J^{-1}]^{(1)}$ can be derived by inverting (21) and taking the coefficient of $\alpha_s^{(4)}/(4\pi)$. $\xi_{45}^{(1)}$ is the coefficient of $\alpha_s^{(4)}/(4\pi)$ in (20), and $Z_\alpha^{(1)}$ the one of $\alpha_s/(4\pi)$ in (14). $[Z_J^{-1}]^{(2)}$ is defined to subtract the remaining pole parts in $F_1^{(2)} + F_1^{(1)}\{\,[Z_J^{-1}]^{(1)} + \xi_{45}^{(1)} + Z_\alpha^{(1)}\}$, such that $C_1^{(2)}$ is IR-finite. Note that to evaluate (75) one needs the $\mathcal{O}(\epsilon^2)$ terms of $F_i^{(1)}$ given in Sec. 4.1, since $[Z_J^{-1}]^{(1)}$ contains a $1/\epsilon^2$ singularity.

These coefficients depend on the IR factorization scale, which cancels only in the product of hard, jet- and shape-function factors. In physical applications the factorization scale ranges between $\sqrt{m_b\Lambda_{\mathrm{QCD}}}$ and m_b. To illustrate the size of the new two-loop correction, we therefore evaluate $C_i(u)$ at $\mu = m_b = 4.8\,\mathrm{GeV}$ (blue/dark grey curves) and at $\mu = 1.5\,\mathrm{GeV}$ (orange/light-grey curves). We emphasize that the difference between these two choices is not a theoretical error – it is compensated by a corresponding scale dependence of the convolution $J \star S$. The value of the strong coupling is $\alpha_s^{(5)}(m_b) = \alpha_s^{(4)}(m_b) = 0.22$, and $\alpha_s^{(4)}(1.5\,\mathrm{GeV}) = 0.3753$. Comparison of the dashed

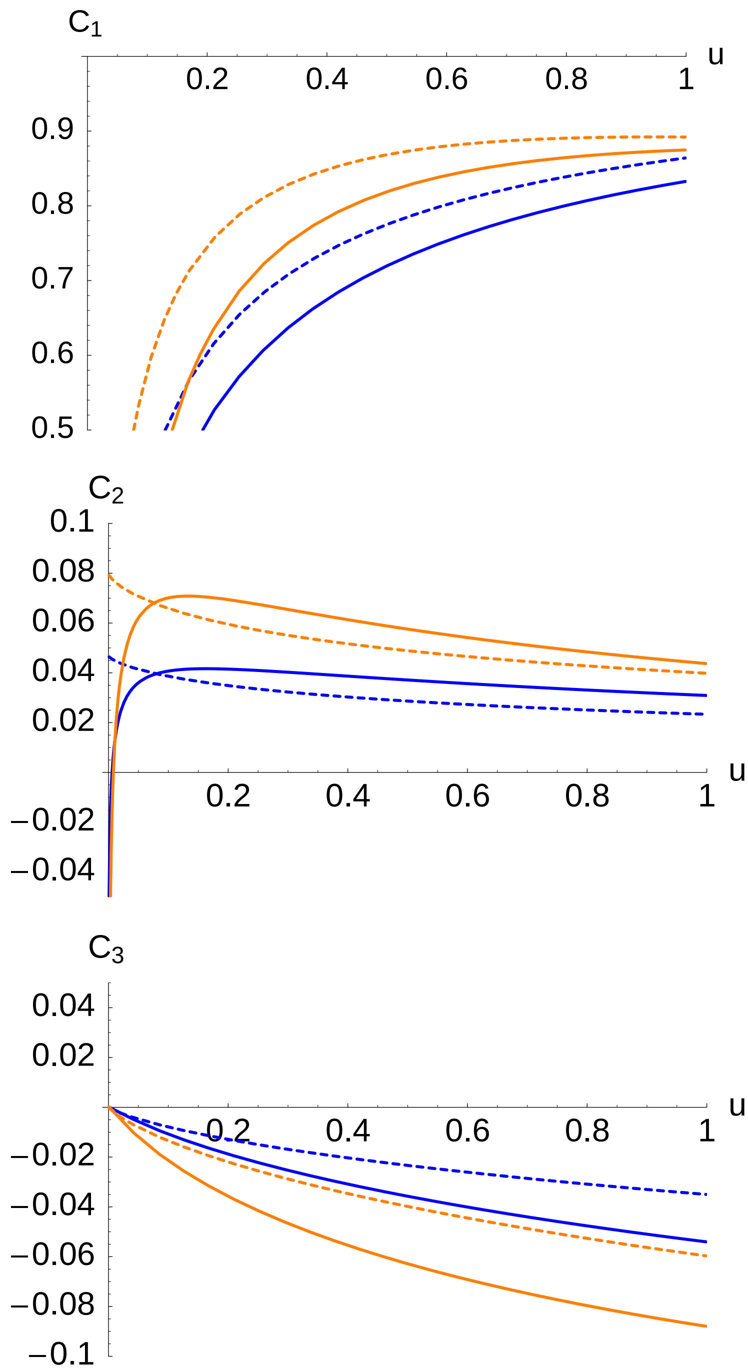

Figure 3: The matching coefficients $C_i(u)$ ($i = 1, 2, 3$ from top to bottom) in the one-loop (dashed) and two-loop (solid) approximation. The blue/dark grey curves refer to $\mu = m_b = 4.8\,\mathrm{GeV}$, the orange/light grey curves to $\mu = 1.5\,\mathrm{GeV}$.

and solid curves of the same colour in Fig. 3 shows that the two-loop corrections are generally very moderate, if not small, except in the region of small u, where increasing powers of large logarithms take over. The implications of this result for the $|V_{ub}|$ determination remain to be investigated. The impact of the $\mathcal{O}(\alpha_s^2)$ terms depends on the combination $C_i C_j \, J \star S$, and the numerical size of the two-loop correction to the jet and shape function has not yet been analyzed. A straightforward evaluation of the partonic structure functions W_i in the shape-function region indicates sizeable two-loop effects. A reanalysis of existing $B \to X_u \ell \bar{\nu}$ decay distribution data with $\mathcal{O}(\alpha_s^2)$ accuracy taking into account renormalization group summation and a model of the B meson shape function is therefore well motivated.

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A Master integrals

The two-loop calculation gives rise to 18 master integrals, which are depicted in Fig. 4. All necessary master integrals were already computed in [31], which chooses a slightly different basis compared to the present work. Certain individual master integrals can also be found in [36–40]. We find almost perfect agreement with the results in [31]. In the case of the crossed six-line master integral I_g in Fig. 4(g) we improve the numerical accuracy of the boundary condition of the finite part. We therefore give our results for I_g explicitly. Our integration measure reads

$$\int [dk] \equiv \int \frac{d^D k}{(2\pi)^D} \, . \tag{76}$$

We define the prefactor

$$S_\Gamma \equiv \frac{1}{(4\pi)^{D/2}\,\Gamma(1-\epsilon)} \; , \tag{77}$$

and express our results in terms of

$$x \equiv \frac{(p_b - p)^2 + i\eta}{m_b^2} = 1 - u \; . \tag{78}$$

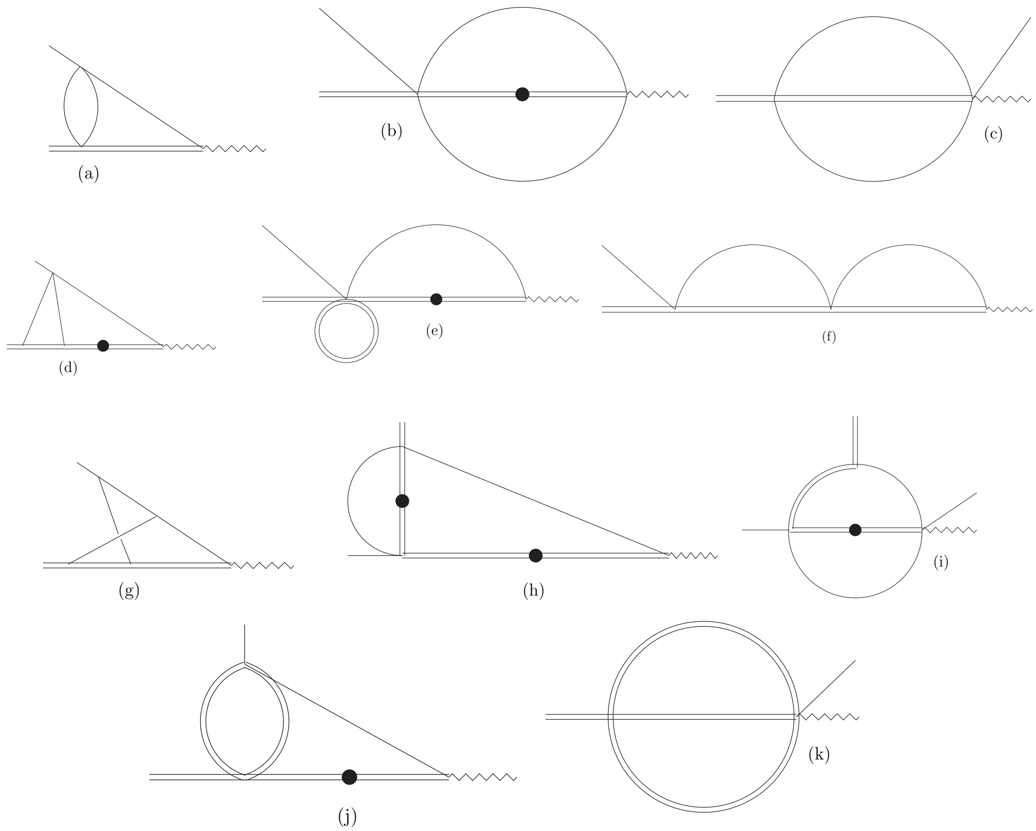

Figure 4: Two-loop master integrals needed for the calculation. Double lines are massive, and single lines are massless. All diagrams stand for scalar integrals with unit numerator. Dots on lines represent squared propagators. Topologies with one or more dots stand for the undotted diagram and all diagrams with *one* single dot, i.e. topology (h) stands for three diagrams.

Here $+i\eta$ stems from the $+i\eta$ prescription which we tacitly assume to be included in the propagators of the integral below. We find

$$\begin{aligned} I_g &= \int [dk_1] \int [dk_2] \, \frac{1}{[(k_2+p_b)^2 - m_b^2]\,(k_2+p)^2\,[(k_1+p_b)^2 - m_b^2]\,(k_2-k_1+p)^2} \\ &\qquad\qquad \times \frac{1}{(k_2-k_1)^2\,k_1^2} \\ &= -S_\Gamma^2\,(m_b^2)^{-2-2\epsilon} \left\{ \frac{c_g^{(-4)}}{\epsilon^4} + \frac{c_g^{(-3)}}{\epsilon^3} + \frac{c_g^{(-2)}}{\epsilon^2} + \frac{c_g^{(-1)}}{\epsilon} + c_g^{(0)} \right\} \end{aligned} \tag{79}$$

with

$$c_g^{(-4)} = \frac{1}{12\,(1-x)^2}\,,$$

$$
\begin{aligned}
c_g^{(-3)} &= -\frac{\ln(1-x)}{3\,(1-x)^2}\,, \\
c_g^{(-2)} &= \frac{1}{72\,(1-x)^2}\left[48\ln^2(1-x) - 5\pi^2\right]\,, \\
c_g^{(-1)} &= \frac{1}{36\,(1-x)^2}\left[-32\ln^3(1-x) + 10\pi^2\ln(1-x) - 267\,\zeta_3\right]\,, \\
c_g^{(0)} &= \frac{1}{(1-x)^2}\left[\frac{8}{9}\ln^4(1-x) - \frac{5}{9}\pi^2\ln^2(1-x) + 8\ln(1-x)\,\mathrm{Li}_3(x)\right. \\
&\qquad\qquad \left. + \frac{65}{3}\ln(1-x)\,\zeta_3 + 4\,\mathrm{Li}_2^2(x) + c_0\right]\,. \qquad (80)
\end{aligned}
$$

The constant c_0, obtained with the package MB.m [25] from a three-dimensional Mellin-Barnes representation, equals $-60.2493267(10)$, where the number in parenthesis gives the uncertainty of the last two digits displayed. The number excludes $c_0 = -89\pi^4/144$ which was found in [31][2]. It suggests $c_0 = -167\pi^4/270$, in agreement with the proposal in [10]. To date there does, however, not exist a value for c_0 that is derived completely analytically. Except for c_0, we obtained all other terms in the master integrals by analytical steps, that is, without fitting rational numbers to numerical values.

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